A CHARACTERIZATION OF TWO-WEIGHT INEQUALITIES FOR A VECTOR-VALUED OPERATOR

JAMES SCURRY

Abstract. We give a characterization of the two-weight inequality for a simple vector-valued operator. Special cases of our result have been considered before in the form of the weighted Carleson embedding theorem, the dyadic positive operators of Nazarov, Treil, and Volberg [4] in the square integrable case, and Lacey, Sawyer, Uriarte-Tuero [2] in the L^p case. The main technique of this paper is a Sawyer-style argument and the characterization is for $1 < p < \infty$. We are unaware of instances where this operator has been given attention in the two-weight setting before.

1. Introduction and Statement of Main Results

Throughout this paper, by a 'weight' ω, we will mean a function $\omega \in L^1_{\text{loc}}(\mathbb{R}^d)$ which satisfies $\omega > 0$. In harmonic analysis, 'weighted theory' or 'weighted norm inequalities' typically refers to the study of an operator's continuity properties when considered as acting on functions from $L^r(\mu)$ to $L^p(\omega)$, where μ and ω are fixed weights and $1 < p, r < \infty$. Specifically, the problem is to give necessary and sufficient conditions on the weights for an operator $T : L^r(\mu) \to L^p(\omega)$ to be bounded. Within weighted theory there is a stark bifurcation; namely, it is divided between one-weight theory and two-weight theory. As suggested by the terminology, one-weight refers to the case where $\mu = \omega$ and two-weight is a reference to the case when μ and ω are different. For most classical operators from harmonic analysis, the one-weight theory is well understood, but the two-weight case is more difficult and complicated than the one-weight case. Due to the work of Dr. Eric Sawyer on the two-weight inequality for fractional integrals [6] and the maximal function [7], there is a standard method for characterizing the two-weight inequality via testing conditions. Explicitly, given an operator $T : L^r(\mu) \to L^p(\omega)$, we test the following inequality

$$\|T(f\omega)\|_{L^p(\omega)} \leq \|f\|_{L^r(\mu)} \tag{1.1}$$

over all f in some special, usually simpler, class of functions. Moreover, save for a few 'simple' operators, it is also often necessary to test the same collection of functions over the dual of an operator, or an appropriate analogue (see [6] and [2]). In deference to Dr. Sawyer, we will refer to these types of testing conditions as 'Sawyer-type testing condtions'.

Further, we will restrict ourselves to the study of a vector-valued operator. As far as we are aware, our operator has previously been unexamined in the two-weight setting, and the closest analogue which has been given attention in the weighted context is the square function. The square function was studied in [8] and [4]; however, the material presented in these papers relies on techniques largely unrelated to those used in this paper. We will instead draw inspiration from [2]; but, prior to introducing our main result, we will dispense with perfunctory notational considerations.

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In the remainder of this paper, σ and w denote fixed weights and $1 < p, r, q < \infty$ will be fixed such that $1 < r \le p < \infty$. Let $\mathcal{D}$ be a dyadic grid in $\mathbb{R}^d$ and $\mathcal{Q}$ a fixed collection of subcubes from $\mathcal{D}$. We take $\tau = \{\tau_Q : Q \in \mathcal{Q}\}$ to be a collection of non-negative constants indexed by the cubes in $\mathcal{Q}$, and denote by $\overline{T}_{\mathcal{Q},\tau} : L^r(\sigma) \to L^p(w)$ the sublinear operator which is given by

$$\overline{T}_{\mathcal{Q},\tau}(f\sigma) = \left(\sum_{R\in\mathcal{Q}} |\tau_R \mathbb{E}_R(f\sigma)\mathbf{1}_R(x)|^q \right)^{\frac{1}{q}}$$

where $f \in L^r(\sigma)$ and $\mathbb{E}_Q(g) = \frac{1}{|Q|}\int_Q g$ for $g \in L^1_{\text{loc}}(\mathbb{R}^d)$. The set $B_{(\mathbb{R}^d,\ell^{q'}(\mathcal{Q}))}$ will indicate the class of all sequences of functions $\{g_Q(x)\}_{Q\in\mathcal{Q}}$ such that

i. $\left(\sum_{Q\in\mathcal{Q}} |g_Q(x)|^{q'} \right)^{\frac{1}{q'}} = 1$, almost every $x \in \mathbb{R}^d$

ii. supp $g_Q \subset Q$ all $Q \in \mathcal{Q}$,

and for $g = \{g_Q\}_{Q\in\mathcal{Q}}$ a sequence of functions on $\mathbb{R}^d$ and $s \in L^1_{\text{loc}}(\mathbb{R}^d)$, gs means $gs = \{g_Q s\} = \{g_Q(x)s(x)\}_{Q\in\mathcal{Q}}$. Finally, given $f \in L^r(\sigma)$, define $a_f = \{(\mathbb{E}_Q(f\sigma)\tau_Q)^{q-1}\mathbf{1}_Q\}_{Q\in\mathcal{Q}}(\overline{T}_{\mathcal{Q},\tau}(f\sigma))^{\frac{-1}{q'}}$ and note that $a_f \in B_{(\mathbb{R}^d,\ell^{q'}(\mathcal{Q}))}$.

The primary contribution of this paper is to characterize the boundedness of $\overline{T}_{\mathcal{Q},\tau}$ in the two-weight setting using Sawyer-type testing conditions. We are motivated by the work of Nazarov, Treil, and Volberg [4] and Lacey, Sawyer, and Uriarte-Tuero [2] who considered operators having the form

$$R(f\sigma) = \sum_{Q\in\mathcal{Q}} \tau_Q \mathbb{E}_Q(f\sigma)\mathbf{1}_Q. \tag{1.2}$$

The operators from (1.2) are characterized in the $L^2(\sigma) \to L^2(w)$ case in [4] and in [2] the $L^r(\sigma) \to L^p(w)$ case, with $1 < r \le p < \infty$, is considered. To be precise, the main result from [2] is

Theorem 1.3 (Lacey, Sawyer, Uriarte-Tuero, [2]). *The operator $R(\cdot\sigma) : L^r(\sigma) \to L^p(w)$ is bounded if and only if*

$$\|\mathbf{1}_K R(\mathbf{1}_K w)\|_{L^{r'}(\sigma)} \lesssim w(K)^{\frac{1}{p'}} \tag{1.4}$$

$$\|\mathbf{1}_K R(\mathbf{1}_K \sigma)\|_{L^p(w)} \lesssim \sigma(K)^{\frac{1}{r}} \tag{1.5}$$

for all $K \in \mathcal{D}$.

The above testing conditions may appear somewhat atypical as they involve only the operator R and there seems to be no mention of a 'dual' operator, but we can easily place them within a more conventional framework: in (1.5) we test the 'original' operator $R(\cdot\sigma)$ over indicator functions of cubes, and in (1.4) we test the 'dual' $R(\cdot w)$ of $R(\cdot\sigma)$ over all such functions.

The first obstacle to our desired characterization for $\overline{T}_{\mathcal{Q},\tau}$ is that the operator $\overline{T}_{\mathcal{Q},\tau}$ *not* linear. We resolve this difficulty by defining a vector-valued operator $T_{\mathcal{Q},\tau}(\cdot\sigma) : L^r(\sigma) \to L^p_{\ell^q}(w)$ which satisfies

$$T_{\mathcal{Q},\tau}(\sigma f) = \{\mathbb{E}_Q(f\sigma)\tau_Q\mathbf{1}_Q\}_{Q\in\mathcal{Q}}.$$

Then $\overline{T}_{\mathcal{Q},\tau}$ and $T_{\mathcal{Q},\tau}$ are related by the following:

$$\int_{\mathbb{R}^d} \overline{T}_{\mathcal{Q},\tau}(f\sigma)^p w = \int_{\mathbb{R}^d} \|T_{\mathcal{Q},\tau}(f\sigma)\|_{\ell^{q'}(\mathcal{Q})}^p w = \int_{\mathbb{R}^d} \langle T_{\mathcal{Q},\tau}(f\sigma), a_f w\rangle_{\ell^q(\mathcal{Q})}.$$

The operator $T_{\mathcal{Q},\tau}$ has a dual which will be denoted by $U_{\mathcal{Q},\tau}$; further, we have $U_{\mathcal{Q},\tau}(\cdot w) : L^{p'}_{\ell^{q'}}(w) \to L^{r'}(\sigma)$ is given by

$$U_{\mathcal{Q},\tau}(\{g_Q w\}_{Q\in\mathcal{Q}})(x) = \sum_{Q\in\mathcal{Q}} \mathbb{E}_Q(g_Q w)\tau_Q \mathbf{1}_Q(x)$$

for $\{g_Q\}_{Q\in\mathcal{Q}} \in L^{p'}_{\ell^{q'}(\mathcal{Q})}(w)$. With $U_{\mathcal{Q},\tau}$ defined, we are able to state the main theorem:

Theorem 1.6. *Let $1 < r \le p < \infty$. Then the operator $\overline{T}_{\mathcal{Q},\tau}$ is bounded from $L^r(\sigma) \to L^p(w)$ if and only if the following two testing conditions hold:*

$$\mathcal{L} := \sup_{Q\in\mathcal{D}} \sup_{\{a_R\}\in B_{(\mathbb{R}^d,\ell^{q'}(\mathcal{Q}))}} w(Q)^{-\frac{r'}{p'}} \int_Q \left(U_{\mathcal{Q},\tau}\left(\{\mathbf{1}_{Q\cap R} a_R\ w\}_{R\in\mathcal{Q}}\right)\right)^{r'} \sigma < \infty \tag{1.7}$$

$$\mathcal{L}_* := \sup_{Q\in\mathcal{D}} \sigma(Q)^{-\frac{p}{r}} \int_Q \overline{T}_{\mathcal{Q},\tau}(\mathbf{1}_Q\sigma)^p w < \infty \tag{1.8}$$

In particular, it will be shown that $\|\overline{T}(\cdot\sigma)\|_{L^r(\sigma)\to L^p(w)} \approx \max\{\mathcal{L}^{\frac{1}{r'}}, \mathcal{L}_*^{\frac{1}{p}}\}$.

A few comments about Theorem 1.6 are in order. First, in the case $r = q = p$ characterizing $\overline{T}$ reduces to a simple application of the Carleson embedding theorem. Namely, $\overline{T}$ is bounded if and only if

$$\sup_{Q\in\mathcal{D}} \frac{1}{\sigma(Q)} \sum_{\substack{R\in\mathcal{Q}\\ R\subset Q}} \frac{w(R)\sigma(R)^p\tau_R}{|R|^p} \lesssim 1$$

and the testing condition on $U_{\mathcal{Q},\tau}$ is superfluous. Secondly, we emphasize that in the case $q = 1$ we obtain a characterization of the operator from (1.2) considered in [2].

Note: Throughout the remainder of this paper, the dependence of the operators on τ and $\mathcal{Q}$ will be suppressed. Further, the notation $A \lesssim B$ will be used to mean $A \le KB$ for some absolute constant K; likewise $A \gtrsim B$ will mean $AK \ge B$ for some absolute constant K. Finally, $A \approx B$ will indicate $A \lesssim B$ and $B \lesssim A$.

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2. Proof of Theorem 1.6: Necessity

Here we prove the necessity of the testing conditions. We suppose that $\overline{T}$ is a bounded operator. The necessity of (1.8) is immediate by taking $f = \mathbf{1}_Q$ for $Q \in \mathcal{D}$ so we only need

to verify the necessity of the conditions on U. Fix $R \in \mathcal{D}$; then for a sequence $\{a_Q\}_{Q\in\mathcal{Q}} \in B_{(\mathbb{R}^d,\ell^{q'}(\mathcal{Q}))}$, we have $\{a_Q\mathbf{1}_R\}_{Q\in\mathcal{Q}} \in L^{p'}_{\ell^{q'}}(w)$. Moreover,

$$\begin{aligned}\left\| U\left(\{a_Q\mathbf{1}_R w\}_{Q\in\mathcal{Q}}\right)\right\|_{L^{r'}(\sigma)} &= \left(\int_{\mathbb{R}^d} \left|U\left(\{a_Q\mathbf{1}_R w\}_{Q\in\mathcal{Q}}\right)\right|^{r'}\sigma\right)^{\frac{1}{r'}}\\ &= \int_{\mathbb{R}^d} U\left(\{a_Q\mathbf{1}_R w\}_{Q\in\mathcal{Q}}\right)h\sigma\end{aligned}$$

where h is an appropriate function from $L^r(\sigma)$ satisfying $\|h\|_{L^r(\sigma)} = 1$. Then using the triangle inequality and duality

$$\begin{aligned}\int_{\mathbb{R}^d} U\left(\{a_Q\mathbf{1}_R w\}_{Q\in\mathcal{Q}}\right)h\sigma &\leq \int_{\mathbb{R}^d} U\left(\{|a_Q\mathbf{1}_R w|\}_{Q\in\mathcal{Q}}\right)|h|\,\sigma\\ &= \int_{\mathbb{R}^d} \langle T(|h|\,\sigma), \{|a_Q\mathbf{1}_R|\}_{Q\in\mathcal{Q}}\rangle_{\ell^q(\mathcal{Q})}.\end{aligned}$$

Now we apply Hölder's inequality in $\ell^q(\mathcal{Q}) - \ell^{q'}(\mathcal{Q})$ and obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \langle T(|h|\,\sigma), \{|a_Q\mathbf{1}_R|\}_{Q\in\mathcal{Q}}\rangle_{\ell^q(\mathcal{Q})} &\leq \int_R \overline{T}(|h|\,\sigma)w\\ &\leq \|\overline{T}(|h|\,\sigma)\|_{L^p(w)} w(R)^{\frac{1}{p'}}\\ &\leq \|\overline{T}(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)} w(R)^{\frac{1}{p'}}.\end{aligned}$$

Hence,

$$\int_R U\left(\{a_Q\mathbf{1}_R w\}_{Q\in\mathcal{Q}}\right)^{r'}\sigma \leq \int_R \left|U\left(\{a_Q\mathbf{1}_R w\}_{Q\in\mathcal{Q}}\right)\right|^{r'}\sigma \leq \|\overline{T}(\cdot\sigma)\|^{r'}_{L^r(\sigma)\to L^p(w)} w(R)^{\frac{r'}{p'}}$$

where $\{a_Q\}_{Q\in\mathcal{Q}} \in B_{(\mathbb{R}^d,\ell^{q'}(\mathcal{Q}))}$ is arbitrary. Taking a supremum we have

$$\sup_{\{a_Q\}\in B_{(\mathbb{R}^d,\ell^{q'}(\mathcal{Q}))}} w(R)^{-\frac{r'}{p'}} \int_R U\left(\{\mathbf{1}_R a_Q\ w\}_{Q\in\mathcal{Q}}\right)^{p'}\sigma \leq \|\overline{T}(\cdot\sigma)\|^{r'}_{L^r(\sigma)\to L^p(w)} \tag{2.1}$$

with the constant independent of $R \in \mathcal{D}$. Taking the supremum over all $R \in \mathcal{D}$ in (2.1) gives the result.

3. Proof of Theorem 1.6: Sufficiency

3.1. **Overview.** In this portion of the paper, the sufficiency of the testing conditions is demonstrated using a Sawyer-style argument; a less complicated version of this type of argument can be found in [2]. Briefly, we review the highlights of such an approach. First, weak-type estimates for the operators U and $\overline{T}$ are obtained which allow for the strengthening of both testing conditions. Subsequently, a point-wise estimate on $\overline{T}$ is achieved and this permits a localization procedure, i.e. we are able to restrict $f\sigma$ to a particular cube. Thereafter, the integral will be broken up in to three parts S_1, S_2, and S_3. Estimates for S_1 and S_2 will follow easily. The strengthened testing conditions will be used in concert with a corona decomposition and a combinatorial argument to estimate S_3. Now we proceed to the proof of the sufficiency of the testing conditions.

3.2. **Technical Considerations.** Here, we collect many of the lemmas which will be needed in the proof of the sufficiency. Recall that we aim to show the operator norm of the sublinear operator $\overline{T}$ is bounded; further, $\overline{T}$ is non-negative. Hence, it is clear that when considering $\int_{\mathbb{R}^d} \overline{T}(f\sigma)^p w$, f can be taken to be a non-negative bounded function with compact support; throughout the remainder of the paper, we will make such an assumption.

We will work with the sets $\Omega_k = \{\overline{T}(f\sigma) > 2^k\}$, which are open since $\overline{T}(f\sigma)(x)$ is lower semi-continuous, and start with a covering lemma.

Lemma 3.1. *(Whitney Decomposition) For each k there exists a collection $\mathcal{Q}_k$ of disjoint cubes satisfying:*

$$\Omega_k = \bigcup_{Q\in\mathcal{Q}_k} Q, \tag{3.2}$$

$$Q^{(1)} \subset \Omega_k\,,\ \ Q^{(2)} \cap \Omega_k^c \neq \emptyset, \tag{3.3}$$

$$\sum_{Q\in\mathcal{Q}_k} \mathbf{1}_{Q^{(1)}} \lesssim \mathbf{1}_{\Omega_k}, \tag{3.4}$$

$$\sup_{Q\in\mathcal{Q}_k} \sharp\{Q' \in \mathcal{Q}_k \ :\ Q' \cap Q^{(1)} \neq \emptyset\} \lesssim 1\,, \tag{3.5}$$

$$Q \in \mathcal{Q}_k\,,\ \ Q' \in \mathcal{Q}_l\,,\ \ Q \subsetneqq Q' \qquad k > l\,. \tag{3.6}$$

where $Q^{(1)}$ is the parent of Q in the dyadic grid, and $Q^{(j+1)} = (Q^{(j)})^{(1)}$ for $1 \leq j$.

Remark: The proof of this lemma can be found in [2, pages 8-9]*, and further, we will reference* (3.3) *as the 'Whitney condition' throughout the remainder of this paper.*

3.2.1. *Obtaining weak-type estimates and strengthening the testing conditions.* We now introduce a lemma which will give us a weak-type characterization for the operators U and $\overline{T}$:

Lemma 3.7. *Assuming* (1.8) *and* (1.7) *hold, for $\{g_Q\}_{Q\in\mathcal{Q}} \in L^{p'}_{\ell^{q'}(\mathcal{Q})}(w)$ and $f \in L^r(\sigma)$, we have*

$$\|U(\{g_Q w\}_{Q\in\mathcal{Q}})\|_{L^{r',\infty}(\sigma)} \ \lesssim\ \mathcal{L}_*^{\frac{1}{p}} \|\{g_Q\}_{Q\in\mathcal{Q}}\|_{L^{p'}_{\ell^{q'}(\mathcal{Q})}(w)}, \tag{3.8}$$

$$\|\overline{T}(f\sigma)\|_{L^{p,\infty}(w)} \ \lesssim\ \mathcal{L}^{\frac{1}{r'}} \|f\|_{L^r(\sigma)}. \tag{3.9}$$

Proof. We will argue the case for (3.8) first. Fix a sequence $\{g_Q\}_{Q\in\mathcal{Q}} \in L^{p'}_{\ell^{q'}}(w)$ and begin by defining $\Gamma_\alpha = \{x : U(\{g_Q w\}_{Q\in\mathcal{Q}}) > \alpha\}$ for $\alpha > 0$. $U(\{g_Q\}_{Q\in\mathcal{Q}} w)(x)$ is lower semi-continuous and so Γ_α is open. Similar to Lemma 3.1, we will perform a Whitney-style decomposition; specifically, for fixed α, let $\{L_j^\alpha\}_{j\in\mathbb{N}}$ be the dyadic cubes from $\mathcal{D}$ which are maximal with respect to the following two conditions:

(i) $L_j^\alpha \cap \Gamma_{2\alpha} \neq \emptyset$,
(ii) $L_j^\alpha \subset \Gamma_\alpha$ all $j \in \mathbb{N}$.

First, we aim to put ourselves in a position to use the testing condition on $\overline{T}$:

$$\begin{aligned}
\sum_{j\in\mathbb{N}}\left(\sigma(L_j^\alpha)^{-1}\int_{L_j^\alpha}U(\{g_Qw\}_{Q\in\mathcal{Q}})\sigma\right)^{r'}\sigma(L_j^\alpha) &= \sum_{j\in\mathbb{N}}\left(\sigma(L_j^\alpha)^{-1}\int_{L_j^\alpha}\langle \mathbf{1}_{L_j^\alpha}\sigma, U\left(\{g_Qw\}_{Q\in\mathcal{Q}}\right)\rangle_{\ell^q(\mathcal{Q})}\right)^{r'}\sigma(L_j^\alpha)\\
&= \sum_{j\in\mathbb{N}}\left(\sigma(L_j^\alpha)^{-1}\int_{L_j^\alpha}\langle T(\mathbf{1}_{L_j^\alpha}\sigma), \{g_Qw\}_{Q\in\mathcal{Q}}\rangle_{\ell^q(\mathcal{Q})}\right)^{r'}\sigma(L_j^\alpha). \qquad (3.10)
\end{aligned}$$

We continue by applying the triangle inequality and Hölder's inequality to obtain

$$\begin{aligned}
(3.10) &\leq \sum_{j\in\mathbb{N}}\left(\sigma(L_j^\alpha)^{-1}\int_{L_j^\alpha}\overline{T}(\mathbf{1}_{L_j^\alpha}\sigma)\|\{g_Qw\}_{Q\in\mathcal{Q}}\|_{\ell^{q'}(\mathcal{Q})}\right)^{r'}\sigma(L_j^\alpha)\\
&= \sum_{j\in\mathbb{N}}\left(\int_{L_j^\alpha}\overline{T}(\mathbf{1}_{L_j^\alpha}\sigma)\|\{g_Qw\}_{Q\in\mathcal{Q}}\|_{\ell^{q'}(\mathcal{Q})}\right)^{r'}\sigma(L_j^\alpha)^{1-r'}\\
&\leq \sum_{j\in\mathbb{N}}\left(\int_{L_j^\alpha}\|\{g_Q\}_{Q\in\mathcal{Q}}\|_{\ell^{q'}(Q)}^{p'}w\right)^{\frac{r'}{p'}}\left(\int_{L_j^\alpha}\overline{T}(\mathbf{1}_{L_j^\alpha}\sigma)^pw\right)^{\frac{r'}{p}}\sigma(L_j^\alpha)^{1-r'}\\
&\lesssim \mathcal{L}_*^{\frac{r'}{p}}\left(\sum_{j\in\mathbb{N}}\int_{L_j^\alpha}\|\{g_Q\}_{Q\in\mathcal{Q}}\|_{\ell^q(\mathcal{Q})}^{p'}w\right)^{\frac{r'}{p'}}\\
&\lesssim \mathcal{L}_*^{\frac{r'}{p}}\|\{g_Q\}_{Q\in\mathcal{Q}}\|_{L^{p'}_{\ell^{q'}(\mathcal{Q})}(w)}^{r'}. \qquad (3.11)
\end{aligned}$$

At this point we will appeal to a 'good-lambda' trick. In particular, we fix α and $\epsilon = 2^{-r'-1} > 0$; further, we define $\mathcal{E} = \{j : \sigma(L_j^\alpha\cap\Gamma_{2\alpha}) < \epsilon\sigma(L_j^\alpha)\}$. So,

$$\begin{aligned}
(2\alpha)^{r'}\sigma(\Gamma_{2\alpha}) &\lesssim \epsilon(2\alpha)^{r'}\sum_{j\in\mathcal{E}}\sigma(L_j^\alpha) + \epsilon^{-1}\sum_{j\notin\mathcal{E}}(2\alpha)^{r'}\sigma(L_j^\alpha)\\
&\leq \epsilon(2\alpha)^{r'}\sum_{j\in\mathcal{E}}\sigma(L_j^\alpha) + \sum_{j\notin\mathcal{E}}2^{-1}(\alpha\sigma(L_j^\alpha)\sigma(L_j^\alpha)^{-1})^{r'}\sigma(L_j^\alpha)\\
&\leq \epsilon(2\alpha)^{r'}\sum_{j\in\mathcal{E}}\sigma(L_j^\alpha) + \sum_{j\notin\mathcal{E}}2^{-1}\left(\sigma(L_j^\alpha)^{-1}\int_{L_j^\alpha}U(\{g_Q\}_{Q\in\mathcal{Q}}w)\right)^{r'}\sigma(L_j^\alpha)\\
&\lesssim \epsilon(2\alpha)^{r'}\sum_{j\in\mathcal{E}}\sigma(L_j^\alpha) + 2^{-1}\mathcal{L}_*^{\frac{r'}{p}}\|\{g_Q\}_{Q\in\mathcal{Q}}\|_{L^{p'}_{\ell^{q'}(\mathcal{Q})}(w)}^{r'}
\end{aligned}$$

where the final inequality follows from (3.11). Hence

$$\begin{aligned}
(2\alpha)^{r'}\sigma(\Gamma_{2\alpha}) &\lesssim 2^{-1}(\alpha)^{r'}\sigma(\Gamma_\alpha) + 2^{-1}\mathcal{L}_*^{\frac{r'}{p}}\|\{g_Q\}_{Q\in\mathcal{Q}}\|_{L^{p'}_{\ell^{q'}(\mathcal{Q})}(w)}^{r'}\\
&\leq 2^{-1}\|U(\{g_Qw\}_{Q\in\mathcal{Q}})\|_{L^{p',\infty}(\sigma)}^{r'} + 2^{-1}\mathcal{L}_*^{\frac{r'}{p}}\|\{g_Q\}_{Q\in\mathcal{Q}}\|_{L^{p'}_{\ell^{q'}}(w)}^{r'}
\end{aligned}$$

which gives (3.8).

Now we consider (3.9). The argument will be similar to that for (3.8). Fix a postive

function $f \in L^r(\sigma)$ and let $\Psi_\alpha = \{x : \overline{T}(f\sigma)(x) > \alpha\}$ for $\alpha > 0$. Again, we perform a Whitney-style decomposition; explicitly, let $\{P_j^\alpha\}_{j\in\mathbb{N}}$ be the dyadic cubes from $\mathcal{D}$ which are maximal with respect to:

(i.) $P_j^\alpha \cap \Psi_{2\alpha} \neq \emptyset$
(ii.) $P_j^\alpha \subset \Psi_\alpha$ all $j \in \mathbb{N}$.

We define $a_f = \overline{T}(f\sigma)^{q-1}(\overline{T}(f\sigma))^{\frac{-1}{q}}$ and attempt to place ourselves in a position where we may use the testing condition on U:

$$\begin{aligned}\sum_{j\in\mathbb{N}}\left(w(P_j^\alpha)^{-1}\int_{P_j^\alpha}\overline{T}(f\sigma)w\right)^p w(P_j^\alpha) &= \sum_{j\in\mathbb{N}}\left(w(P_j^\alpha)^{-1}\int_{P_j^\alpha}\langle \mathbf{1}_{P_j^\alpha} w a_f, T(f\sigma)\rangle_{\ell^q(\mathcal{Q})}\right)^p w(P_j^\alpha)\\ &= \sum_{j\in\mathbb{N}}\left(w(P_j^\alpha)^{-1}\int_{P_j^\alpha}U(\mathbf{1}_{P_j^\alpha} w a_f)f\sigma\right)^p w(P_j^\alpha) \qquad (3.12)\end{aligned}$$

Now by Hölder's inequality,

$$\begin{aligned}(3.12) &\leq \sum_{j\in\mathbb{N}}\left(\int_{P_j^\alpha}U(a_f\mathbf{1}_{P_j^\alpha}w)^{r'}\sigma\right)^{\frac{p}{r'}}\left(\int_{P_j^\alpha}f^r\sigma\right)^{\frac{p}{r}} w(P_j^\alpha)^{1-p}\\ &\lesssim \sum_{j\in\mathbb{N}}\mathcal{L}^{\frac{p}{r'}}\left(\int_{P_j^\alpha}f^r\sigma\right)^{\frac{p}{r}}\\ &\lesssim \mathcal{L}^{\frac{p}{r'}}\|f\|^p_{L^r(\sigma)}.\end{aligned}$$

As before we use a 'good-lambda' trick; we fix α and $\epsilon = 2^{-p-1}$. Further, define $\Upsilon = \{j : w(P_j^\alpha \cap \Psi_{2\alpha}) < \epsilon w(P_j^\alpha)\}$. So

$$\begin{aligned}(2\alpha)^p w(\Psi_{2\alpha}) &\lesssim \epsilon(2\alpha)^p\sum_{j\in\Upsilon}w(P_j^\alpha) + \epsilon^{-1}\sum_{j\notin\Upsilon}(2\alpha)^p w(P_j^\alpha)\\ &\lesssim \epsilon(2\alpha)^p\sum_{j\in\Upsilon}w(P_j^\alpha) + 2^{-1}\sum_{j\notin\Upsilon}(\alpha w(P_j^\alpha)w(P_j^\alpha)^{-1})^p w(P_j^\alpha)\\ &\lesssim \epsilon(2\alpha)^p\sum_{j\in\Upsilon}w(P_j^\alpha) + 2^{-1}\sum_{j\notin\Upsilon}\left(w(P_j^\alpha)^{-1}\int_{P_j^\alpha}\overline{T}(f\sigma)w\right)^p w(P_j^\alpha)\\ &\lesssim \epsilon(2\alpha)^p\sum_{j\in\Upsilon}w(P_j^\alpha) + 2^{-1}\mathcal{L}^{\frac{p}{r'}}\|f\|^p_{L^r(\sigma)}\\ &\leq \epsilon(2\alpha)^p w(\Psi_\alpha) + 2^{-1}\mathcal{L}^{\frac{p}{r'}}\|f\|^p_{L^r(\sigma)}\end{aligned}$$

Now we have

$$\begin{aligned}(2\alpha)^p w(\Psi_{2\alpha}) &\lesssim 2^{-1}\alpha^p w(\Psi_\alpha) + 2^{-1}\mathcal{L}^{\frac{p}{r'}}\|f\|^p_{L^r(\sigma)}\\ &\leq 2^{-1}\|\overline{T}(f\sigma)\|^p_{L^{p,\infty}(w)} + \mathcal{L}^{\frac{p}{r'}}\|f\|^p_{L^r(\sigma)}\end{aligned}$$

and this gives (3.9). □

A consequence of Lemma 3.7 is that we can make slight modifications to the testing conditions on $\overline{T}$ and U:

Lemma 3.13. *For each $Q \in \mathcal{D}$ and for any positive $\{a_R\}_{R\in\mathcal{Q}} \in B_{\mathbb{R}^d,\ell^{q'}(\mathcal{Q})}$, we have*

$$\int_{\mathbb{R}^d} \overline{T}(\mathbf{1}_Q\sigma)^p w \;\lesssim\; \mathcal{L}_*\sigma(Q)^{\frac{p}{r}}, \tag{3.14}$$

$$\int_{\mathbb{R}^d} U\left(\{\mathbf{1}_Q a_R w\}_{R\in\mathcal{Q}}\right)^{r'} \;\lesssim\; \mathcal{L}w(Q)^{\frac{r'}{p'}}. \tag{3.15}$$

Remark 3.16. The integrals above are distinguished from the integrals in (1.8) and (1.7) by integration over $\mathbb{R}^d$ as opposed to Q.

Proof. First we will show the case for (3.14). By (3.8), we know $\|U(\cdot w)\|_{L^{p'}_{\ell^{q'}(\mathcal{Q})}(w)\to L^{r',\infty}(\sigma)} \lesssim \mathcal{L}_*^{\frac{1}{p}}$; therefore by duality, we have for each $f \in L^{r,1}(\sigma)$,

$$\|\overline{T}(f\sigma)\|_{L^p(w)} \lesssim \mathcal{L}_*^{\frac{1}{p}}\|f\|_{L^{r,1}(\sigma)}.$$

Since for all $Q \in \mathcal{D}$, $\mathbf{1}_Q \in L^{r,1}(\sigma)$ and $\|\mathbf{1}_Q\|_{L^{r,1}(\sigma)} = \sigma(Q)^{\frac{1}{r}}$, we have

$$\|\overline{T}(\mathbf{1}_Q\sigma)\|_{L^p(w)} \lesssim \mathcal{L}_*^{\frac{1}{p}}\sigma(Q)^{\frac{1}{r}}$$

which gives the desired result.

We conclude by verifying (3.15) holds. Consider, for $\{a_R\}_{R\in\mathcal{Q}}$ and Q as above,

$$\left(\int_{\mathbb{R}^d} U(\{\mathbf{1}_Q\{a_R\}_{R\in\mathcal{Q}})^{r'} w\right)^{\frac{1}{r'}} \sigma \;=\; \int_{\mathbb{R}^d} U(\{\mathbf{1}_Q\{a_R\}_{R\in\mathcal{Q}} w) h\sigma$$

for some $h \in L^r(\sigma)$. Then using duality and Hölder's inequality in $\ell^q(\mathcal{Q}) - \ell^{q'}(\mathcal{Q})$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} U(\{\mathbf{1}_Q\{a_R\}_{R\in\mathcal{Q}} w) h\sigma \;&=\; \int_{\mathbb{R}^d} \langle \mathbf{1}_Q a_R, T(h\sigma)\rangle_{\ell^q(\mathcal{Q})} w \\ &\le\; \int_Q \overline{T}(h\sigma) w. \end{aligned}\tag{3.17}$$

A bit of notation: by $(\overline{T}(h\sigma))^*$ and $(\mathbf{1}_Q)^*$, we will mean the symmetric decreasing rearrangements of $\overline{T}(h\sigma)$ and $\mathbf{1}_Q$ with respect to w. With this new notation in hand, we continue from (3.17) by applying Hölder's inequality and using (3.9) to obtain

$$\begin{aligned}(3.17) \;&\le\; \int_{\mathbb{R}} (\overline{T}(h\sigma))^* \,(\mathbf{1}_Q)^*\, w \\ &\le\; \|\overline{T}(h\sigma)\|_{L^{p,\infty}(w)} w(Q)^{\frac{1}{p'}} \\ &\le\; \|\overline{T}(\cdot\sigma)\|_{L^r(\sigma)\to L^{p,\infty}(w)} w(Q)^{\frac{1}{p'}} \\ &\lesssim\; \mathcal{L}^{\frac{1}{r'}} w(Q)^{\frac{1}{p'}}.\end{aligned}$$

The foregoing inequalities yield

$$\int_{\mathbb{R}^d} U(\mathbf{1}_Q\{a_R\}_{R\in\mathcal{Q}} w)^{r'}\sigma \le \mathcal{L}w(Q)^{\frac{r'}{p'}}$$

and we are done. $\square$

Now we introduce some notation which will permit us a slight bit of concision when stating subsequent lemmas. For a fixed cube $Q \in \mathcal{D}$, we can split $\overline{T}_{\mathcal{Q},\tau}$ into operators $\overline{T}^{\text{in}}_{\mathcal{Q},\tau,Q}$ and $\overline{T}^{\text{out}}_{\mathcal{Q},\tau,Q}$ defined by

$$\overline{T}^{\text{in}}_{\mathcal{Q},\tau,Q}(f\sigma) = \left(\sum_{\substack{R \subseteq Q \\ R \in \mathcal{Q}}} \left| \mathbb{E}_R\left(f\sigma\right) \tau_R \mathbf{1}_R \right|^q \right)^{\frac{1}{q}},$$

$$\overline{T}^{\text{out}}_{\mathcal{Q},\tau,Q}(f\sigma) = \left(\sum_{\substack{Q \subset R \\ R \in \mathcal{Q}}} \left| \mathbb{E}_R\left(f\sigma\right) \tau_R \mathbf{1}_R \right|^q \right)^{\frac{1}{q}}.$$

We note the inequality

$$\overline{T}_{\mathcal{Q},\tau}(f\sigma)(x) \le \overline{T}^{\text{in}}_{\mathcal{Q},\tau,Q}(f\sigma)(x) + \overline{T}^{\text{out}}_{\mathcal{Q},\tau,Q}(f\sigma)(x)$$

for all x and $Q \in \mathcal{D}$. At this point we consider a mainstay of the Sawyer-type argument: the maximum principal.

Lemma 3.18. *For all k and $Q \in \mathcal{Q}_k$ we have*

$$\max\left\{ \overline{T}^{\text{out}}_{Q^{(1)}}(\mathbf{1}_{Q^{(2)}} f\sigma)(x)\,,\, \overline{T}(\mathbf{1}_{(Q^{(2)})^c} f\sigma)(x) \right\} \le 2^k\,, \qquad x \in Q\,.$$

Proof. By Lemma 3.1, there is $z \in Q^{(2)} \cap \Omega_k^c$. Thus for $x \in Q$ we have

$$\overline{T}(\mathbf{1}_{(Q^{(2)})^c} f\sigma)(x) = \overline{T}^{\text{out}}_{Q^{(1)}}(\mathbf{1}_{(Q^{(2)})^c} f\sigma)(x) \le \overline{T}(f\sigma)(z) \le 2^k\,.$$

□

Define

$$E_k(Q) = Q \cap (\Omega_{k+2} - \Omega_{k+3})\,, \qquad Q \in \mathcal{Q}_k\,.$$

for all k. Then the following lemma follows from Lemma 3.18,

Lemma 3.19. *For all k and $x \in E_k(Q)$, we have*

$$2^k \le \overline{T}^{\text{in}}_{Q^{(1)}}(\mathbf{1}_{Q^{(1)}} f\sigma)(x).$$

Proof. By Lemma 3.18 and the sub-linearity of $\overline{T}$, we have for $x \in E_k(Q)$

$$\begin{aligned} 2^{k+2} - 2^{k+1} &\le \overline{T}(f\sigma)(x) - \overline{T}^{\text{out}}_{Q^{(1)}}(\mathbf{1}_{Q^{(1)}} f\sigma)(x) - \overline{T}(\mathbf{1}_{(Q^{(1)})^c} f\sigma)(x) \\ &\le \overline{T}^{\text{in}}_{Q^{(1)}}(\mathbf{1}_{Q^{(1)}} f\sigma)(x). \end{aligned}$$

Noting $2^{k+2} - 2^{k+1} \le 2^k$, we obtain $2^k \le \overline{T}^{\text{in}}_{Q^{(1)}}(\mathbf{1}_{Q^{(1)}} f\sigma)(x)$. □

3.3. The terms S_1, S_2, and S_3.

Later, we will break up the integral

$$\int_{\mathbb{R}^d} \overline{T}(f\sigma)^p w$$

into level sets; here we estimate the size of $w(E_k(Q))$ for fixed k and $Q \in \mathcal{Q}_k$:

$$\begin{aligned} 2^k w(E_k(Q)) &\leq \int_{E_k(Q)} \overline{T}(\mathbf{1}_{Q^{(1)}} f\sigma) w \\ &= \int_{E_k(Q)} \|T(\mathbf{1}_{Q^{(1)}} f\sigma)\|_{\ell^q(Q)} w. \end{aligned} \tag{3.20}$$

Now passing from the operator $T(\cdot\sigma)$ to the operator U via duality and noting that in so doing we move to an operator with a 'simpler' argument, namely $\mathbf{1}_{E_k(Q)} a_f w$, we obtain

$$\begin{aligned} (3.20) &= \int_{Q^{(1)}} \langle \{f\mathbf{1}_P\}_{P\in\mathcal{Q}}, \{\mathbb{E}(\mathbf{1}_{E_k(Q)} a_P w)\tau_P\}_{P\in\mathcal{Q}}\rangle_{\ell^q(\mathcal{Q})} \sigma \\ &= \int_{Q^{(1)}} f \cdot U\left(\{\mathbf{1}_{E_k(Q)} a_P w\}_{P\in\mathcal{Q}}\right) \sigma \end{aligned} \tag{3.21}$$

where $a_f = T(f\sigma)^{q-1}(\overline{T}(f\sigma))^{\frac{-1}{q}} = \{a_P(x)\}_{P\in\mathcal{Q}} \in B_{(\mathbb{R}^d, \ell^{q'}(\mathcal{Q}))}$. Define

$$\begin{aligned} \alpha_k(Q) &= \int_{Q^{(1)}\setminus\Omega_{k+3}} fU\left(\{a_P \mathbf{1}_{E_k(Q)} w\}_{P\in\mathcal{Q}}\right)\sigma, \\ \beta_k(Q) &= \int_{Q^{(1)}\cap\Omega_{k+3}} fU\left(\{a_P \mathbf{1}_{E_k(Q)} w\}_{P\in\mathcal{Q}}\right)\sigma \end{aligned}$$

so that (3.21) becomes $\alpha_k(Q) + \beta_k(Q)$. Then

$$\begin{aligned} \int_{\mathbb{R}^d} \|T(f\sigma)\|^p_{\ell^q(\mathcal{Q})} w &\leq 2^{3p} \sum_{k=-\infty}^{\infty} 2^{kp} w(\Omega_{k+2} - \Omega_{k+3}) \\ &= 2^{3p} \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathcal{Q}_k} 2^{kp} w(E_k(Q)). \end{aligned} \tag{3.22}$$

At this point we introduce a parameter $0 < \eta < 1$ (whose value will be specified later) and define

$$\begin{aligned} \mathcal{Q}_k^1 &= \{Q \in \mathcal{Q}_k \,:\, w(E_k(Q)) \leq \eta w(Q)\}, \\ \mathcal{Q}_k^2 &= \{Q \in \mathcal{Q}_k \,:\, w(E_k(Q)) > \eta w(Q),\ \alpha_k(Q) > \beta_k(Q)\}, \\ \mathcal{Q}_k^3 &= \mathcal{Q}_k - \mathcal{Q}_k^1 - \mathcal{Q}_k^2. \end{aligned} \tag{3.23}$$

Then (3.22) becomes $2^{3p}\sum_{j=1}^3 S_j$ where

$$S_j = \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathcal{Q}_k^j} 2^{kp} w(E_k(Q)), \qquad j = 1, 2, 3,$$

Now we consider S_1, S_2, and S_3 separately.

3.4. **Estimating S_1 and S_2.**

First we will examine S_1. By the definition of the set $\mathcal{Q}_k^1$ we have

$$S_1 \lesssim \eta \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathcal{Q}_k^1} 2^{kp} w(Q) \leq \eta \sum_{k=-\infty}^{\infty} 2^{kp} w(\Omega_k) \lesssim \eta \|\overline{T}(f\sigma)\|^p_{L^p(w)}. \tag{3.24}$$

We fix $0 < \eta < 1$; since η has been fixed such that $\eta < 1$, we may incorporate $\eta\|\overline{T}(\sigma f)\|^p_{L^p(w)}$ into $\|\overline{T}(\sigma f)\|^p_{L^p(w)}$. Consequently, we now have the desired estimate for S_1.

At this point we estimate S_2. Using the definition of $\mathcal{Q}^2_k$:

$$\eta 2^k w(Q) \le 2^k w(E_k(Q)) \lesssim \alpha_k(Q) = \int_{Q^{(1)}\setminus\Omega_{k+3}} fU(\{\mathbf{1}_{E_k(Q)}a_P w\}_{P\in\mathcal{Q}})\sigma.$$

We apply Hölder's inequality and use Lemma 3.13 to obtain

$$\begin{aligned}\int_{Q^{(1)}\setminus\Omega_{k+3}} fU\left(\{(\mathbf{1}_{E_k(Q)}a_P w\}_{P\in\mathcal{Q}}\right)\sigma &\le \left[\int_{Q^{(1)}\setminus\Omega_{k+3}} f^r\ \sigma\right]^{\frac{1}{r}} \\ &\quad \left[\int_Q \left(U(\{\mathbf{1}_{E_k(Q)}a_P w\}_{P\in\mathcal{Q}})\right)^{r'}\ \sigma\right]^{\frac{1}{r'}} \\ &\le \mathfrak{L}^{\frac{1}{r'}}\left[\int_{Q^{(1)}\setminus\Omega_{k+3}} f^r\ \sigma\right]^{\frac{1}{r}}\cdot w(Q)^{\frac{1}{p'}}.\end{aligned}$$

Consequently,

$$2^k \lesssim \mathfrak{L}^{\frac{1}{r'}}\eta^{-1}w(Q)^{-\frac{1}{p}}\left[\int_{Q^{(1)}\setminus\Omega_{k+3}} f^r\ \sigma\right]^{\frac{1}{r}}$$

so that

$$\begin{aligned}S_2 &= \sum_{k=-\infty}^{\infty}\sum_{Q\in\mathcal{Q}^1_k} 2^{kp}w(E_k(Q)) \\ &\lesssim \eta^{-p}\mathfrak{L}^{\frac{p}{r'}}\sum_{k=-\infty}^{\infty}\sum_{Q\in\mathcal{Q}^2_k}\frac{w(E_k(Q))}{w(Q)}\left[\int_{Q^{(1)}\setminus\Omega_{k+3}} f^r\ \sigma\right]^{\frac{p}{r}} \\ &\lesssim \eta^{-p}\mathfrak{L}^{\frac{p}{r'}}\left[\int_{\mathbb{R}^d} f^r\sum_{k=-\infty}^{\infty}\sum_{Q\in\mathcal{Q}^2_k}\mathbf{1}_{Q^{(1)}\setminus\Omega_{k+3}}\ \sigma\right]^{\frac{p}{r}}.\end{aligned}$$

By (3.5), we have

$$\begin{aligned}\eta^{-p}\mathcal{L}^{\frac{p}{r'}}\left[\int_{\mathbb{R}^d} f^r\sum_{k=-\infty}^{\infty}\sum_{Q\in\mathcal{Q}^2_k}\mathbf{1}_{Q^{(1)}\setminus\Omega_{k+3}}\ \sigma\right]^{\frac{p}{r}} &\le \eta^{-p}\mathcal{L}^{\frac{p}{r'}}\left[\int_{\mathbb{R}^d} f^r\sum_{k=-\infty}^{\infty}\mathbf{1}_{\Omega_k\setminus\Omega_{k+3}}\ \sigma\right]^{\frac{p}{r}} \\ &\lesssim \eta^{-p}\mathcal{L}^{\frac{p}{r'}}\left[\int_{\mathbb{R}^d} f^r\ \sigma\right]^{\frac{p}{r}}.\end{aligned}$$

Hence, we have

$$S_2 \lesssim \eta^{-p}\mathcal{L}^{\frac{p}{r'}}\|f\|^p_{L^r(\sigma)} \tag{3.25}$$

which finishes our result for S_2.

3.5. **Estimating S_3.** Now we estimate S_3. First we introduce a theorem from basic harmonic analysis which will frequently be employed in the remainder of this paper:

Theorem 3.26. *Let ω be a weight and $1 < s \le \infty$. For $g \in L^s(\omega)$, define*

$$M_\omega g(x) = \sup_{Q \in \mathcal{D}} \mathbf{1}_Q(x) \mathbb{E}_Q^\omega |g|$$

where $\mathbb{E}_Q^\omega |g| = \omega(Q)^{-1} \int_Q |g| \, \omega$. Then $M_\omega : L^s(\omega) \to L^s(\omega)$ is a bounded operator.

Remark 3.27. The proof of Theorem 3.26 follows by applying a standard covering lemma in the context of dyadic cubes.

3.5.1. *Breaking S_3 up into S_4 and S_5.* For $N \in \mathbb{Z}$ and $0 \le M < 3$, define

$$S_{3,M}^N = \sum_{\substack{k \equiv M \text{ mod } 3 \\ k \ge -N}} \sum_{Q \in \mathcal{Q}_k^3} 2^{kp} w(E_k(Q)) \,.$$

Then

$$S_3 = \lim_{N \to \infty} \sum_{j=0}^{2} S_{3,j}^N \,.$$

The goal here will be to prove an estimate for $S_{3,M}^N$ which is independent of M and N; once such a result is obtained, simply taking the limit of the sums as $N \to \infty$ will then give the desired result.

We make a definition:

$$\mathcal{R}_k(Q) = \{R \in \mathcal{Q}_{k+3} \ : \ Q^{(1)} \cap R \neq \emptyset\} \,, \qquad Q \in \mathcal{Q}_k^3 \,. \tag{3.28}$$

for $Q \in \mathcal{D}$. Intuitively, for $Q \in \mathcal{D}$, (3.28) is the collection of neighbors of Q which lie in Q_{k+3}, and will be a collection of cubes on which $\mathbf{1}_R(x) U(\{a_P \mathbf{1}_{E_k(Q) \cap P} w\}_{P \in \mathcal{Q}})(x)$ is constant; now we formulate these claims explicitly in the following lemma:

Lemma 3.29.

(i.) We have the following equality

$$Q^{(1)} \cap \Omega_{k+3} = \bigcup_{R \in \mathcal{R}_k(Q)} R \,.$$

(ii.) For $Q \in \mathcal{D}$ and $R \in R_k(Q)$, we have $\mathbf{1}_R(x) U(\{a_P \mathbf{1}_{E_k(Q) \cap P} w\}_{P \in \mathcal{Q}})(x)$ is constant for $x \in \mathcal{R}$.

Proof. First we will show *(i.)*. Suppose $Q \in \mathcal{D}$ and $R \in \mathcal{R}_k(Q)$. Then either $R \subsetneq Q^{(1)}$ or $Q^{(1)}$ is contained in R. If $Q^{(1)} \subset R$, then since we are dealing with cubes in a dyadic grid, $Q^{(1)}$ must be contained in R and as a result, $Q^{(2)}$ is also a subset of R. Therefore, $Q^{(2)} \subset \Omega_{k+3}$; but this violates the Whitney decomposition. So it must be that $R \subsetneq Q^{(1)}$.

We consider *(ii.)* at this point. Simply note that since $R^{(1)} \cap E_k(Q) = \emptyset$, we have

$$\mathbf{1}_R(x) U(\{a_P \mathbf{1}_{E_k(Q) \cap P} w\}_{P \in \mathcal{Q}})(x) = \mathbf{1}_R(x) \sum_{\substack{K \in \mathcal{Q} \\ R^{(1)} \subsetneq K}} \tau_K \cdot \mathbb{E}_K (a_K \mathbf{1}_{E_k(Q)} w) \,.$$

$\square$

Now by Lemma (3.29) *(ii.)*, we have

$$\begin{aligned}\beta_k(Q) &\leq \sum_{R\in\mathcal{R}_k(Q)} \int_R f\cdot U(\{a_P \mathbf{1}_{E_k(Q)\cap P} w\}_{P\in\mathcal{Q}})\,\sigma \\ &= \sum_{R\in\mathcal{R}_k(Q)} \int_R U(\{a_P \mathbf{1}_{E_k(Q)\cap P} w\}_{P\in\mathcal{Q}})\,\sigma\cdot \mathbb{E}^\sigma_R f\,.\end{aligned}$$

The following theorem figures prominently throughout the balance of this paper and will be referenced as the Corona Decomposition:

Theorem 3.30 (Corona Decomposition)**.** *For each $0\leq M<3$ and $N\in\mathbb{Z}$, there exists a collection $\mathcal{G}^N_M \subset \bigcup_{\substack{k\equiv M \bmod 3\\ k\geq -N}} \mathcal{Q}_k$ which has the following properties:*

(i.) For each $Q\in \bigcup_{\substack{k\equiv M \bmod 3\\ k\geq -N}} \mathcal{Q}_k$ there is $G\in\mathcal{G}^N_M$ so that $Q\subset G$ and $\mathbb{E}^\sigma_Q f\leq 2\mathbb{E}^\sigma_G f$.

(ii.) If $G, G'\in\mathcal{G}^N_M$, and $G\subsetneq G'$ then $2\mathbb{E}^\sigma_{G'} f<\mathbb{E}^\sigma_G f$.

(iii.) If $\Gamma^N_M(Q)$ is the minimal cube in $\mathcal{G}^N_M$ which contains Q, then $\mathbb{E}^\sigma_Q f\leq 2\mathbb{E}^\sigma_{\Gamma^N_M(Q)} f$.

Remark 3.31. The proof is well-known and can be found in [6, pages 540-541], where the decomposition is referred to as the construction of the 'principal cubes'; more detailed discussions of this construction can be found in [1, pages 53-65] or [3, pages 804-806].

Due to part *(ii)*, in Theorem 3.30, we have the following corollary:

Corollary 3.32. *For $0\leq M<3$ and $N\in\mathbb{Z}$,*

$$\sum_{G\in\mathcal{G}^N_M} \sigma(G)(E^\sigma_G f)^r \lesssim \int_{\mathbb{R}^d} f^r\sigma. \tag{3.33}$$

Proof. Fix M and N. Let Z be a set of measure zero outside which $\overline{T}(f\sigma)$ is finite and fix $x\in\mathbb{R}^d\backslash Z$. The sequence $\{G^x_j\}_{j=1}^\infty$ denotes the collection of all cubes from $\mathcal{G}^N_M$ which contain x. Then

$$\sum_{G\in\mathcal{G}^N_M} \mathbf{1}_G(x)\mathbb{E}^\sigma_G f \;=\; \sum_{j=1}^\infty \mathbb{E}^\sigma_{G^x_j} f.$$

Since $N\in\mathbb{Z}$ and $0\leq M<3$ are fixed and $x\in\mathbb{R}^d\backslash Z$, there is a unique cube $G^x(N)$ among the $\{G^x_j\}_{j\in\mathbb{N}}$ so that $G^x(N)\subset G^x_j$ all $j\in\mathbb{N}$. By Theorem (3.30) we have

$$\sum_{j=1}^\infty \mathbb{E}^\sigma_{G^x_j} f \lesssim \mathbb{E}^\sigma_{G^x(N)} f\leq M_\sigma f(x).$$

Now from Theorem 3.26 we obtain the result since

$$\sum_{G\in G^N_M} \mathbf{1}_G(x)(\mathbb{E}^\sigma_G f)^r \leq \left(\sum_{G\in G^N_M} \mathbf{1}_G(x)\mathbb{E}^\sigma_G f\right)^r \lesssim M_\sigma f(x)^r.$$

□

We make another definition: for fixed $Q \in \mathcal{D}$,

$$\mathcal{N}_k(Q) = \{Q' \in \mathcal{Q}_k \ : \ Q' \cap Q^{(1)} \neq \emptyset\}.$$

Therefore, for a fixed $Q \in \mathcal{D}$, $\mathcal{N}_k(Q)$ are 'neighbors' of Q which reside in $\mathcal{Q}_k$, while $\mathcal{R}_k$ consists of 'neighbors' which are in $\mathcal{Q}_{k+3}$. Most importantly, for any cube in $\mathcal{Q}_k$, $\sharp\mathcal{N}_k(Q)$ is uniformly bounded by (3.5). For a fixed cube $Q \in \mathcal{Q}_k$, there is an important property of $\mathcal{R}_k(Q)$ which will be exploited momentarily; namely, if $0 \leq M < 3$ and $N \in \mathbb{Z}$ are fixed with $k \equiv M \mod 3$ and $P \in \mathcal{R}_k(Q)$ such that $P \subset Q$, then either $\Gamma_M^N(Q) = \Gamma_M^N(P)$ or $P \in \mathcal{G}_M^N$. Now for a fixed cube we consider 'neighbors' of 'neighbors' i.e., we will consider sums of the following nature:

$$\begin{aligned}
\beta_{k,4}(Q) &= \sum_{Q' \in \mathcal{N}_k(Q)} \sum_{\substack{R \in \mathcal{R}_k(Q) \\ \Gamma_M^N(R) = \Gamma_M^N(Q') \\ R \subset Q'}} \int_R U(\{a_P \mathbf{1}_{Q \cap P} w\}_{P \in \mathcal{Q}})\, \sigma \cdot \mathbb{E}_R^\sigma f \,, \\
\beta_{k,5}(Q) &= \sum_{Q' \in \mathcal{N}_k(Q)} \sum_{\substack{R \in \mathcal{R}_k(Q) \\ R \in \mathcal{G}_M^N}} \int_R U(\{a_P \mathbf{1}_{Q \cap P} w\}_{P \in \mathcal{Q}})\, \sigma \cdot \mathbb{E}_R^\sigma f \,,
\end{aligned}$$

and the import of the above is that we have the following estimate

$$\beta_k(Q) \leq \beta_{k,4}(Q) + \beta_{k,5}(Q).$$

By the definition (3.23), we have

$$\eta 2^k w(Q) \leq 2^k w(E_k(Q)) \leq \beta_k(Q)\,,$$

which implies

$$2^k \lesssim \frac{\beta_k(Q)}{\eta w(Q)}\,. \tag{3.34}$$

A consequence of (3.34) is for $0 \leq M < 3$ and $N \in \mathbb{Z}$ we can estimate $S_{3,M}^N$ as follows:

$$\begin{aligned}
S_{3,M}^N &\lesssim \eta^{-p}\big[S_{4,M} + S_{5,M}\big] \\
S_{v,M}^N &= \sum_{\substack{k \geq -N \\ k \equiv M \mod 3}} \sum_{Q \in \mathcal{Q}_k^3} \frac{w(E_k(Q))}{w(Q)^p} \beta_{k,v}(Q)^p \qquad v = 4, 5\,.
\end{aligned}$$

Moreover, we will define

$$S_v = \lim_{N \to \infty} \sum_{M=0}^{2} \sum_{\substack{k \geq -N \\ k \equiv M \mod 3}} S_{v,M}^N \qquad v = 4, 5\,.$$

3.5.2. *Estimating S_4.* First we consider S_4. Fix $0 \leq M < 3$ and $N \in \mathbb{Z}$. For the remainder of this section, we deal only with integers k which satisfy $k \equiv M \mod 3$. For a fixed cube $Q \in \mathcal{Q}_k^3$, we estimate $\beta_{k,4}(Q)^p$ in the following way

$$\beta_{k,4}(Q)^p \lesssim \sum_{Q' \in \mathcal{N}_k(Q)} \Bigg[\sum_{\substack{R \in \mathcal{R}_k(Q) \\ \Gamma_M^N(R) = \Gamma_M^N(Q') \\ R \subset Q'}} \int_R U(\{\mathbf{1}_{Q \cap P} w\}_{P \in \mathcal{Q}})\, \sigma \cdot \mathbb{E}_R^\sigma f \Bigg]^p \tag{3.35}$$

and note that (3.35) is justified by the uniform bound which we have on $\sharp\mathcal{N}_k(Q)$. For a fixed $G \in \mathcal{G}_M^N$ and fixed Q, define

$$S'_{k,4}(Q,G) = \frac{w(E_k(Q))}{w(Q)^p} \sum_{\substack{Q' \in \mathcal{N}_k(Q) \\ \Gamma_M^N(Q')=G}} \Bigg[\sum_{\substack{R \in \mathcal{R}_k(Q) \\ \Gamma_M^N(R)=G \\ R \subset Q'}} \int_R U(\{a_P \mathbf{1}_{Q\cap P} w\}_{P\in\mathcal{Q}})\, \sigma \cdot \mathbb{E}_R^\sigma f \Bigg]^p .$$

Now we estimate $S'_{k,4}(Q,G)$ for fixed Q and G:

$$\begin{aligned} S'_{k,4}(Q,G) &\lesssim (\mathbb{E}_G^\sigma f)^p w(E_k(Q)) \sum_{Q' \in \mathcal{N}_k(Q)} \Bigg[w(Q)^{-1} \sum_{\substack{R \in \mathcal{R}_k(Q) \\ \Gamma_M^N(R)=G \\ R \subset Q'}} \int_R U(\{a_P \mathbf{1}_{Q\cap P} w\}_{P\in\mathcal{Q}})\, \sigma \Bigg]^p \\ &\lesssim (\mathbb{E}_G^\sigma f)^p w(E_k(Q)) \sum_{Q' \in \mathcal{N}_k(Q)} \Big[w(Q)^{-1} \int_{Q'} U(\{a_P \mathbf{1}_{Q\cap P} w\}_{P\in\mathcal{Q}})\, \sigma \Big]^p \\ &= (\mathbb{E}_G^\sigma f)^p w(E_k(Q)) \Big[w(Q)^{-1} \int_Q \langle T(\mathbf{1}_G\sigma), \{\mathbf{1}_{Q\cap P} w a_P\}_{P\in\mathcal{Q}} \rangle_{\ell^q(\mathcal{Q})}\, w \Big]^p . \end{aligned}$$

Recall $\|a_Q(x)\|_{\ell^q(Q)} = 1$ for almost every $x \in \mathbb{R}^d$, and apply Hölder's inequality in $\ell^q(\mathcal{Q}) - \ell^{q'}(\mathcal{Q})$ to obtain

$$\begin{aligned} &(\mathbb{E}_G^\sigma f)^p w(E_k(Q)) \Big[w(Q)^{-1} \int_Q \langle T(\mathbf{1}_G\sigma), \{\mathbf{1}_{Q\cap P} w a_P\}_{P\in\mathcal{Q}} \rangle_{\ell^q(\mathcal{Q})}\, w \Big]^p \\ &\qquad\qquad \leq (\mathbb{E}_G^\sigma f)^p w(E_k(Q)) \Big[w(Q)^{-1} \int_Q \overline{T}(\mathbf{1}_G\sigma) w \Big]^p . \end{aligned}$$

An important observation is the following: the sum above is part of a linearization of the maximal function M_w. Specifically, $\overline{T}\,(\mathbf{1}_G\sigma)$ is a function in $L^p(w)$, so we may use Theorem 3.26 to estimate $\|M_w(\overline{T}(\mathbf{1}_G\sigma))\|_{L^p(w)} \lesssim \|\overline{T}(\mathbf{1}_G\sigma)\|_{L^p(w)}$. Hence, for fixed $G \in \mathcal{G}_M^N$,

$$\begin{aligned} \sum_k \sum_{Q\in\mathcal{Q}_k} S'_{k,4}(Q,G) &\leq (\mathbb{E}_G^\sigma f)^p \sum_k \sum_{Q\in\mathcal{Q}_k} w(E_k(Q)) \Big[w(Q)^{-1} \int_Q \overline{T}(\mathbf{1}_G\sigma)\, w \Big]^p \\ &\lesssim (\mathbb{E}_G^\sigma f)^p \int_{\mathbb{R}^d} M_w(\overline{T}(\mathbf{1}_G\sigma))^p\, w \\ &\lesssim (\mathbb{E}_G^\sigma f)^p \int_{\mathbb{R}^d} \overline{T}(\mathbf{1}_G\sigma)^p\, w . \end{aligned}$$

We recall Lemma 3.13 and continue with:

$$(\mathbb{E}_G^\sigma f)^p \int_{\mathbb{R}^d} \overline{T}(\mathbf{1}_G\sigma)^p\, w \lesssim \mathfrak{L}_*(\mathbb{E}_G^\sigma f)^p \sigma(G)^{\frac{p}{r}} .$$

Now summing over all G we have

$$
\begin{aligned}
S_{4,M}^N &= \sum_{G\in\mathcal{G}_M^N}\sum_k\sum_{Q\in\mathcal{Q}_k} S'_{k,4}(Q,G)\\
&\lesssim \mathfrak{L}_*\sum_{G\in\mathcal{G}_M^N}(\mathbb{E}_G^\sigma f)^p\sigma(G)^{\frac{p}{r}}\\
&\lesssim \mathfrak{L}_*\Big[\sum_{G\in\mathcal{G}_M^N}(\mathbb{E}_G^\sigma f)^r\sigma(G)\Big]^{\frac{p}{r}}. \qquad (3.36)
\end{aligned}
$$

At this point we note that (3.36) gives a discretization of $\|M_\sigma f\|_r^p$; hence we can immediately conclude an estimate for $S_{4,M}^N$ with

$$
\mathfrak{L}_*\Big[\sum_{G\in\mathcal{G}_M^N}(\mathbb{E}_G^\sigma f)^r\sigma(G)\Big]^{\frac{p}{r}}\lesssim \mathfrak{L}_*\|M_\sigma f\|_{L^r(\sigma)}^p\lesssim \mathfrak{L}_*\|f\|_{L^r(\sigma)}^p\,.
$$

Since the above bound is independent of M and N, it is clear S_4 is finite and, moreover,

$$
S_4\lesssim \mathcal{L}_*\|f\|_{L^r(\sigma)}^p \qquad (3.37)
$$

3.5.3. *Estimating S_5.* Finally, we estimate S_5. Again, as in the case of estimating S_4, we fix $0\le M<3$ and $N\in\mathbb{Z}$; further, for the remainder of the section, we also assume all integers k satisfy $k\equiv M \bmod 3$. Recall

$$
\beta_{k,5}=\sum_{Q'\in\mathcal{N}_k(Q)}\sum_{\substack{R\in\mathcal{R}_k(Q)\\ R\in\mathcal{G}_M^N}}\int_R U(\{a_P\mathbf{1}_{Q\cap P}w\}_{P\in\mathcal{Q}})\,\sigma\cdot\mathbb{E}_R^\sigma f\,,
$$

and consider we have

$$
\frac{w(E_k(Q))}{w(Q)^p}\Bigg[\sum_{\substack{R\in\mathcal{R}_k(Q)\\ R\in\mathcal{G}_M^N}}\int_R U(\{a_P\mathbf{1}_{Q\cap P}w\}_{P\in\mathcal{Q}})\,\sigma\cdot\mathbb{E}_R^\sigma f\Bigg]^p\lesssim
$$

$$
\frac{w(E_k(Q))}{w(Q)^p}\Bigg[\sum_{\substack{R\in\mathcal{R}_k(Q)\\ R\in\mathcal{G}_M^N}}\sigma(R)^{-\frac{r'}{r}}\bigg(\int_R U(\{\mathbf{1}_P w a_P\}_{P\in\mathcal{Q}})\,\sigma\bigg)^{r'}\Bigg]^{\frac{p}{r'}}\Bigg[\sum_{\substack{R\in\mathcal{R}_k(Q)\\ R\in\mathcal{G}_M^N}}\sigma(R)\cdot(\mathbb{E}_R^\sigma f)^r\Bigg]^{\frac{p}{r}}
$$

from multiplying and dividing by $\sigma(G)^{\frac{1}{r}}$ and subsequently applying Hölder's inequality. In the interest of clarity, let

$$
\beta_{k,5,1}(Q)=\frac{w(E_k(Q))}{w(Q)^p}\Bigg[\sum_{\substack{R\in\mathcal{R}_k(Q)\\ R\in\mathcal{G}_M^N}}\sigma(R)^{-\frac{r'}{r}}\bigg(\int_R U\left(\{\mathbf{1}_{P\cap R}a_P w\}_{P\in\mathcal{Q}}\right)\,\sigma\bigg)^{r'}\Bigg]^{\frac{p}{r'}},
$$

$$
\beta_{k,5,2}(Q)=\Bigg[\sum_{\substack{R\in\mathcal{R}_k(Q)\\ R\in\mathcal{G}_M^N}}\sigma(R)\cdot(\mathbb{E}_R^\sigma f)^r\Bigg]^{\frac{p}{r}}.
$$

The expression $\beta_{k,5,1}(Q)$ is bounded above by $\mathfrak{L}^{\frac{p}{r'}}$. Indeed,

$$\beta_{k,5,1}(Q) \lesssim w(Q)^{p-1}\Bigg[\sum_{\substack{R\in\mathcal{R}_k(Q)\\ R\in\mathcal{G}_M^N}} \sigma(R)^{-\frac{r'}{r}}\left(\int_R U\left(\{\mathbf{1}_{P\cap R}a_P w\}_{P\in\mathcal{Q}}\right)\ \sigma\right)^{r'}\Bigg]^{\frac{p}{r'}} \tag{3.38}$$

and after applying Hölder's inequality to (3.38) we obtain

$$\begin{aligned}(3.38) \ &\lesssim\ w(Q)^{p-1}\Bigg[\sum_{\substack{R\in\mathcal{R}_k(Q)\\ R\in\mathcal{G}_M^N}}\left(\int_R U\left(\{\mathbf{1}_{P\cap R}a_P w\}_{P\in\mathcal{Q}}\right)^{r'}\ \sigma\right)\Bigg]^{\frac{p}{r'}}\\ &\lesssim\ w(Q)^{p-1}\|U\left(\{\mathbf{1}_Q a_P w\}_{P\in\mathcal{Q}}\right)\|^p_{L^{r'}(\sigma)}\\ &\lesssim\ \mathfrak{L}^{\frac{p}{r'}}\,.\end{aligned}$$

where the last inequality follows from Lemma 3.13.

Now we use (3.5) to again estimate

$$\begin{aligned}S_{5,M}^N &\lesssim \mathfrak{L}^{\frac{p}{r'}}\sum_k\sum_{Q\in\mathcal{Q}_k^3}\beta_{k,5,2}(Q)\\ &\lesssim \mathfrak{L}^{\frac{p}{r'}}\Bigg[\sum_k\sum_{Q\in\mathcal{Q}_k^3}\sum_{\substack{R\in\mathcal{R}_k(Q)\\ R\in\mathcal{G}_M^N}}\sigma(R)\cdot(\mathbb{E}_R^\sigma f)^r\Bigg]^{\frac{p}{r}}\,.\end{aligned}$$

Combinatorial considerations now come to the forefront. Namely, for fixed $R\in\mathcal{D}$, if

$$c(R)=\sharp\{j : R\in\mathcal{R}_j(Q)\text{ some }Q\}.$$

then there is no a priori upper bound on $c(R)$. We overcome this problem with the following theorem

Theorem 3.39. *[Bounded Occurrences of R] Fix a cube R, and for $1\le l\le c(R)$ suppose that*

(i.) *there is an integer k_l and $Q_l\in\mathcal{Q}_{k_l}^3$ with $R\in\mathcal{R}_{k_l}(Q)$,*
(ii.) *the pairs (Q_l,k_l) are distinct.*

We then have that $c(R)\lesssim 1$, with the implied constant depending upon the dimension, and η, the small constant previously mentioned.

Remark 3.40. Since $\overline{T}(f\sigma)w$ is locally integrable, $\overline{T}(f\sigma)$ is finite almost everywhere with respect to Lebesgue measure; therefore, $c(R)$ cannot be infinite, for this would imply $R\subset\{x:\overline{T}(f\sigma)(x)=\infty\}$ and consequently $|R|=0$, which is impossible.

Proof. Fix $R\in\mathcal{D}$ such that there exists $k_1,\dots,k_{c(R)}\in\mathbb{Z}$ and cubes $Q_1,\dots,Q_{c(R)}$ so that $R\in\mathcal{R}_{k_j}(Q_j)$ for all $1\le j\le c(R)$ and the pairs (Q_j,k_j) are distinct. We argue by contradiction that $c(R)\lesssim 1$. The dyadic structure of $\mathcal{D}$ immediately implies that by possibly reording we must have the following

$$Q_1\subseteq Q_2\subseteq\dots\subseteq Q_{c(R)}. \tag{3.41}$$

Then, as previously stated in this paper, we have $R\subset Q_j^{(1)}$ for each j by (3.3). At this point we consider two cases; namely

(a.) $Q_1 \subsetneq Q_2 \subsetneq ... \subsetneq Q_{c(R)}$
(b.) $Q_1 = ... = Q_{c(R)}$.

First we want to inspect case *(a.)* We may assume that $k_1 > ... > k_{c(R)}$ by (3.3) (Whitney condition); also it is clear that case *(a.)* implies

$$R \subset Q_1^{(1)} \subset ... \subset Q_{c(R)}^{(1)}.$$

Hence, by the above and the definition of $\mathcal{R}_{k_1}$ and $\mathcal{R}_{k_{c(R)}}$, $R \in \mathcal{Q}_{k_1+3}$ and $R \in \mathcal{Q}_{k_{c(R)}+3}$. We conclude $R \in \mathcal{Q}_l$ for $k_{c(R)} + 3 \leq l \leq k_1 + 3$. Since we are assuming that $c(R) \lesssim 1$ fails, without loss of generality we may take $c(R) = 7$. Then we have $R, Q_7 \in Q_{k_7}$:

$$\begin{gathered} R \subset Q_1^{(1)} \subsetneq ... \subsetneq Q_7^{(1)} \implies \\ R^{(2)} \subset Q_7^{(1)} \end{gathered}$$

and this contradicts (3.3). Hence, there is a uniform bound on the number of strict inequalities in (3.41), and so we only need to consider *(b.)*

If *(b.)* holds then by (3.23) we have $w(E_{k_j}(Q_1)) > \eta w(Q_1)$ for all $1 \leq j \leq c(R)$. We can without loss of generality assume the k_i are distinct. Then the $E_{k_j}(Q_1)$ are also distinct and

$$w(Q_1) = \sum_{j\in\mathbb{Z}} w(E_j(Q_1)) \geq \sum_{j=1}^{c(R)} w(E_{k_j}(Q_1)) > \sum_{j=1}^{c(R)} w(Q_1)\eta$$

so that it must be $c(R) \leq \eta^{-1}$ and we are done. □

Application of the theorem immediately yields

$$\begin{aligned} \mathfrak{L}^{\frac{p}{r'}} \Bigg[\sum_k \sum_{Q \in \mathcal{Q}_k^3} \sum_{\substack{R \in \mathcal{R}_k(Q) \\ R \in \mathcal{G}_M^N}} \sigma(R) \cdot (\mathbb{E}_R^\sigma f)^r \Bigg]^{\frac{p}{r}} &\lesssim \mathfrak{L}^{\frac{p}{r'}} \Bigg[\sum_{G \in \mathcal{G}_M^N} \sigma(G) (\mathbb{E}_G^\sigma f)^r \Bigg]^{\frac{p}{r}} \\ &\lesssim \mathfrak{L}^{\frac{p}{r'}} \|f\|_{L^r(\sigma)}^p , \end{aligned}$$

where we once again use Theorem 3.26. Now we have obtained a bound on $S_{5,M}^N$ which is independent of M and N; from this we conclude that S_5 is finite and in particular,

$$S_5 \lesssim \mathcal{L}^{\frac{p}{r'}} \|f\|_{L^r(\sigma)}^p. \tag{3.42}$$

We combine the estimates for S_1, S_2, S_4, and S_5 ((3.24), (3.25), (3.37), and (3.42)) to see

$$\|\overline{T}(f\sigma)\|_{L^p(w)} \lesssim (\eta^{\frac{1}{p}} \|\overline{T}(f\sigma)\|_{L^p(w)} + \mathcal{L}^{\frac{1}{r'}} \eta^{-1} + \mathcal{L}_*^{\frac{1}{p}} + \mathcal{L}^{\frac{1}{r'}})$$

which immediately implies $\|\overline{T}(f\sigma)\|_{L^p(w)} \lesssim \max\{\mathcal{L}^{\frac{1}{r'}}, \mathcal{L}_*^{\frac{1}{p}}\}$, provided η is small enough. This completes the proof of Theorem 1.6.

4. Concluding Remarks

It is expected that the main result of this paper extends readily to separable homogeneous spaces. Further, it is also anticipated that the methods used in this paper can be adapted to the setting of more complicated operators such as dyadic Calderón-Zygmund type operators. The author plans to explore both of these ideas in the future.

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School of Mathematics, Georgia Institute of Technology, Atlanta GA 30332
E-mail address: jscurry3@math.gatech.edu